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Study of Near-Optimal Endurance-Maximizing Periodic Cruise Trajectories

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A near-optimal periodic solution to the maximum-endurance cruise problem is investigated. Point-mass models are developed for different types of aircraft. Energy-state-based methods are used to determine minimum-fuel climb and maximum-endurance descent schedules in the altitude-airspeed plane, which are then pieced together with transition arcs to form the periodic cruise solution. A trajectory-tracking controller is designed to make the point-mass models track the periodic cruise trajectories. The tracking controller is designed using the feedback linearization methodology. Closed-loop simulations are then used to compute the fuel consumption resulting from the use of periodic trajectories. These values are compared with the steady-state, optimal-endurance cruise fuel consumption values. For a model of the F/A-18 aircraft, it was found that savings of about 17% could be realized if the engines can be turned off when the aircraft is not on the climb schedule. However, if the throttle cannot be set below flight idle, the periodic cruise trajectory is found to produce worse performance than the steady-state cruise. Simulations with a model of an F-4 in periodic cruise with nonzero minimum throttle did show a modest improvement over the steady-state cruise performance, but only by 2.7%.

Nomenclature

D = drag
E_s = specific energy
g = acceleration of gravity
h = altitude
L = lift
m = mass
T = thrust
V = velocity
w_f = fuel flow rate
α = angle of attack
γ = flight-path angle

I. Introduction

THE cruise segment of a flight forms the dominant portion of most fixed-wing aircraft operations. It has long been known that the steady-state cruise arc can be nonoptimal for maximizing range or endurance [1]. Developmental efforts on digital flight management systems during the 1970s led to a thorough reexamination of this problem [2,3]. Specifically, in a series of papers, Speyer et al. showed that a periodic trajectory can provide better fuel economy when compared with the steady-state cruise arc [4–9]. More recent research has examined the cruise problem using numerical techniques [10–12]. Most of the research efforts showed that the maximum achievable fuel savings in periodic cruise for the *range* optimization problem is less than 5% for conventional aircraft configurations [1–10]. However, in [11,12], Sachs and Christodoulou, using a point-mass model and the multiple-shooting method, showed that a periodic optimal solution to the *endurance* problem (maximum time aloft for a given fuel load) can be substantially higher, on the order of 50%.

A typical endurance-maximizing trajectory starts at the lowest permissible altitude with a fuel-optimal, full-throttle climb. As the aircraft approaches a chosen maximum altitude, the engine power is set to zero or idle and the aircraft transitions onto a maximum-endurance glide path. Maximum power is applied and the aircraft begins its fuel-optimal climb again toward its high-altitude point as soon as it reaches the minimum permissible altitude. Optimal periodic cruise requires the aircraft to repeat this power-on/power-off cycle over and over. When compared with the classical steady-state cruise, the periodic cruise requires the aircraft to continuously accelerate or decelerate throughout its trajectory. The cycle alternates between higher engine efficiency at low-altitude and the lower drag at high-altitude to achieve savings, because neither point by itself is optimal. The endurance problem is of practical importance in surveillance, reconnaissance, and data relay missions. Another use of periodic cruise trajectories is in performing fuel-optimal maneuvers to confuse enemy interceptors.

The present research was motivated by the results of [11,12]. Although the aircraft configuration used in [11,12] was not identified, it appears that realistic airframe and engine models may have been employed. The objective of the present research is to demonstrate a periodic trajectory as a solution to the optimal-endurance problem using a simple approach to construct a near-optimal trajectory. Instead of attacking the high-order, two-point boundary-value problem resulting from the application of optimal control theory [13] to the aircraft point-mass model, the present research seeks to construct near-optimal solutions using the energy-state model [14]. In this approach, a fuel-optimal climb schedule is patched together with an endurance-optimal descent schedule in the altitude-airspeed plane. Both these schedules are constructed using the energy-state model. The transition arcs between the climb and descent schedules are constructed as minimum-energy-loss trajectories. A trajectory-tracking controller is used to make the point-mass model follow the periodic altitude-airspeed trajectory.

The periodic cruise analysis was applied to two different aircraft models. The first model was of the F/A-18 aircraft and the second

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was of the F-4 aircraft. The F-4 model has been used elsewhere in the literature for investigations into periodic cruise and was readily available. The F/A-18 model was available through NASA and is more realistic than the F-4 model and represents a more modern design. For the F/A-18 aircraft, it was found that some improvement in aircraft endurance could be obtained if the engines were allowed to be completely shut off during the transition arcs and the descent schedule. Requiring the aircraft to fly the transition and descent at flight-idle throttle setting completely eliminated this saving. For the F-4 aircraft, minor fuel savings were observed even if the engines are not allowed to be completely shut off.

An additional investigation using the hodograph set was undertaken to examine the potential for fuel savings in employing periodic cruise trajectories. According to optimal control theory, for optimal solutions to exist, the hodograph set or the extended velocity set has to be convex. Classical theory of periodic cruise [4,7,8] asserts that the periodic cruise becomes optimal whenever the hodograph set is nonconvex. It is shown that the hodograph set for the F/A-18 aircraft is nonconvex if the throttle is allowed to transition all the way to zero. If the throttle cannot be moved below flight idle, which is a practical constraint usually in effect, the hodograph set turns out to be convex.

Section II will discuss the two different aircraft models used in the present study, and the design of trajectory-tracking controllers. Section III will present the methods used in determining the steady-state and periodic cruise trajectories, and will present the results of simulations with different aircraft models. Section IV will discuss the hodograph set analysis for the periodic cruise problem that helps explain the results of Sec. III. Section V will present the conclusions from this research and some recommendations for future work.

II. Modeling and Control

This section will discuss the development of the point-mass models for different aircraft, and the aerodynamic and propulsion models used in each. Models for the F/A-18 and the F-4 will be discussed. A nonlinear controller for tracking prescribed trajectories is also developed. This controller is derived using the feedback linearization method, which transforms the nonlinear dynamic model into a linear form. A control is computed for the linear system, and then is transformed back to the original system to give the actual control.

A. Point-Mass Model

The state variables of the point-mass model are altitude, airspeed, and flight-path angle. The point-mass equations of motion are

$$\dot{h} = V \sin\gamma \tag{1}$$

$$\dot{V} = \frac{T\cos\alpha - D}{m} - g\sin\gamma \tag{2}$$

$$\dot{\gamma} = \frac{g}{V}\left(\frac{L + T\sin\alpha}{mg} - \cos\gamma\right) \tag{3}$$

$$\dot{m} = -w_f \tag{4}$$

The control variables are the throttle, which affects T, and angle of attack, which affects L and D (in some studies, the lift coefficient is the control variable, and the drag coefficient is expressed as a function of the lift coefficient, and $\alpha = 0$ [10,11]. The fuel flow rate and thrust are generally functions of altitude, Mach number, and throttle setting. Note that the mass is held constant, although in the simulations to be discussed later, the fuel flow rate is integrated with time to determine the amount of fuel consumed. It is standard assumption that the amount of fuel consumed over the period in question is a small enough fraction of the total aircraft weight that the difference is negligible.

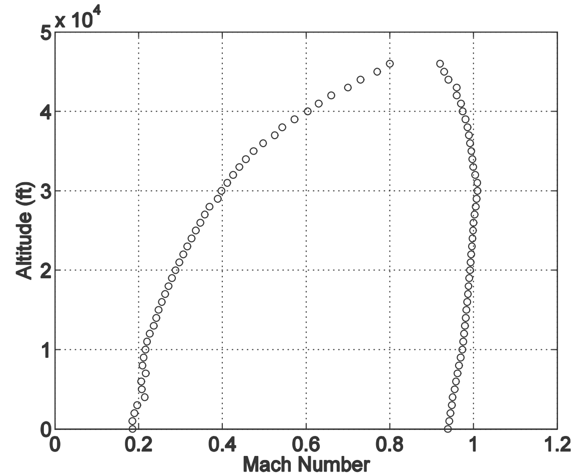

Fig. 1 Level flight envelope for F/A-18.

B. F/A-18 Model

A point-mass model was developed using code from NASA Dryden for the F/A-18 HARV, which uses GE F404-400 engines. The aerodynamics function returns the lift and drag coefficients for the given Mach number, altitude, angle of attack, and angle of sideslip. The propulsion function returns net thrust and fuel flow rate at the given Mach number, altitude, angle of attack, and power lever angle (throttle). The mass was assumed to be 1087.8 slugs (35,000 lbm) and the reference area was 400 ft^2. The level flight envelope computed using the point-mass model is shown in Fig. 1. Note that the flight envelope corresponds to military power.

C. F-4 Model

A model of the McDonnell–Douglas F-4 Phantom II was used in other studies of the range problem [10,15]. In this model, throttle is expressed as a percentage of maximum, and the lift coefficient is the control variable instead of angle of attack. The drag coefficient is defined as a function of the lift coefficient. The aircraft mass was 1164.9 slugs, and the reference area was 530.11 ft^2. It was not specified what the limits of the propulsion model were, but a comparison of the propulsion model with other sources suggested that the maximum throttle corresponds to military power [16,17]. The maximum value of the lift coefficient was not specified, and so a value of 1.0 was assumed. Using this value, the level flight envelope was computed and is shown in Fig. 2. The slightly supersonic maximum speed indicated in Fig. 2 is therefore somewhat

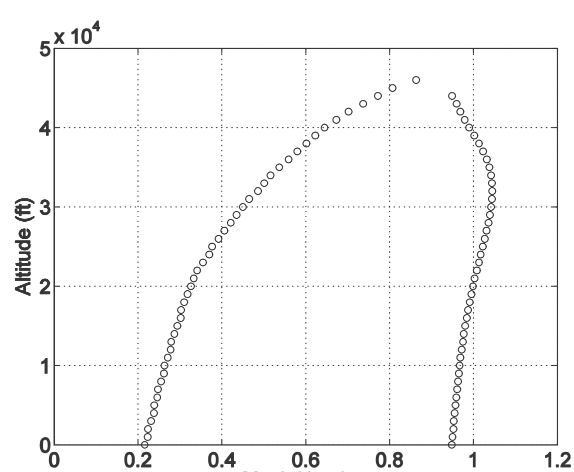

Fig. 2 Level flight envelope for F-4.

unrealistic, but otherwise the envelope is qualitatively similar to the F/A-18 flight envelope shown in Fig. 1.

It should be noted that in this model, the specific fuel consumption at a given altitude and airspeed is independent of the throttle setting. Typically, the thrust specific fuel consumption of a gas turbine increases with decreasing throttle [18], and so this model is somewhat inaccurate.

D. Trajectory-Tracking Controller

As indicated in Sec. I, the energy-state modeling generates periodic cruise trajectories in the altitude-airspeed plane. Thus, a controller is needed to make the point-mass model to follow a prescribed trajectory. In this formalism, the trajectories to be tracked are not expressed as explicit functions of time. Instead, as will be discussed in the following section, the optimal climb and descent profiles are specified as points on the $h - V$ plane at fixed energy levels. A controller was designed using feedback linearization [19] to track a reference altitude command, with the angle of attack as the control variable. Because the throttle is set at fixed values along segments of the periodic cruise trajectory, the speed is controlled by changing the flight-path angle, which is controlled by changing the lift force or the angle of attack. The controller is derived as follows: differentiating Eq. (1) with respect to time gives

$$\ddot{h} = \frac{\partial}{\partial h}(V \sin\gamma)\frac{\mathrm{d}h}{\mathrm{d}t} + \frac{\partial}{\partial V}(V \sin\gamma)\frac{\mathrm{d}V}{\mathrm{d}t} + \frac{\partial}{\partial \gamma}(V \sin\gamma)\frac{\mathrm{d}\gamma}{\mathrm{d}t} \tag{5}$$

Taking the partial derivatives and substituting from Eqs. (1–3),

$$\ddot{h} = 0 + \sin\gamma\left(\frac{T\cos\alpha - D}{m} - g\sin\gamma\right) - V\cos\gamma\frac{g}{V}\left(\frac{L + T\sin\alpha}{mg} - \cos\gamma\right) = \sin\gamma\left(\frac{T\cos\alpha - D}{m}\right) - g + \cos\gamma\left(\frac{L + T\sin\alpha}{m}\right) \tag{6}$$

The preceding equation is written as

$$\ddot{h} = F + G\nu \tag{7}$$

where F, G, and ν are defined by

$$F = \sin\gamma\left(\frac{T\cos\alpha - D}{m}\right) - g \tag{8}$$

$$G = \cos\gamma \tag{9}$$

$$\nu = \frac{L + T\sin\alpha}{m} \tag{10}$$

Now let $F + G\nu = \mu$, where μ is defined by a proportional-integral-derivative feedback control law:

$$\mu = K_P(h_c - h) + K_I\int_o^{tf}(h_c - h)\,\mathrm{d}t + K_D(\dot{h}_c - \dot{h}) \tag{11}$$

where h_c and $\dot{h}_c$ are determined from the periodic cruise schedule and the gains K_P, K_I, K_D are chosen to give good trajectory tracking. Once the linear control is computed, the pseudocontrol ν can be solved for in terms of F and G which can also be computed. Because $\gamma < \pi/2$, G^{-1} always exists. The required value of the actual control variable α is then found by iteratively solving Eq. (10). A controller can also be derived to follow a velocity command, although this requires partial derivatives that must be numerically approximated.

III. Optimal Cruise

In this section, the aircraft models and trajectory controllers described in Sec. II are used to assess the potential fuel savings in employing periodic cruise. The following sections will first discuss the derivation and computation of steady-state and periodic cruise trajectories. The results of simulations with various aircraft types flying periodic cruise trajectories are then presented.

A. Steady-State Cruise

Aircraft normally spend most of a flight at or near some fixed altitude and airspeed. Assuming a steady-state condition, the airspeed and altitude can be optimized for either maximum range or maximum endurance. The conventional wisdom is that the steady-state cruise is the best that can be achieved, other than that as fuel is burned and the aircraft weight changes, so does the cruise point. The steady-state cruise condition therefore serves as the basis for comparison with periodic trajectories.

The maximum-endurance cruise problem is to find the level flight condition that maximizes the time aloft for a given amount of fuel consumed, or equivalently, minimizes the fuel consumption rate, which is a function of airspeed, altitude, and throttle. The maximum-endurance cruise point was found using the MATLAB® optimization toolbox function "fmincon" [20]. The free parameters were h, V, α, and throttle (power lever angle, or PLA), the cost function was the fuel flow rate, and equality constraints were imposed on the sum of the horizontal forces and the sum of the vertical forces, i.e., the trim condition.

B. Periodic Cruise

As discussed in Sec. I, there is a theoretical possibility that a periodic cruise trajectory can provide better cruise performance than the steady-state condition. Instead of using multiple-shooting methods to generate a trajectory as in [11,12], the more intuitive, near-optimal, energy-state methods were employed. It was shown in [14,21] how fuel-optimal climb and maximum-endurance descent paths can be found using energy methods. The energy-state model is a lower-order approximation to the point-mass model in which the states are mass and specific energy:

$$E_s = h + V^2/2g \tag{12}$$

The dynamic equations are then

$$\dot{E}_s = \frac{V(T\cos\alpha - D)}{mg} \tag{13}$$

$$\dot{m} = -w_f \tag{14}$$

Angle of attack is often assumed to be zero for these analyses, but in the present research, α is found by satisfying vertical equilibrium.

1. *Energy-State Method for Minimum-Fuel Climb*

It has been shown in [14,21] that the fuel-optimal climb schedule can be determined using the necessary condition

$$\frac{\partial}{\partial h}\left[\frac{V(T - D)}{\dot{m}}\right]\Bigg|_{E=\mathrm{Const}} = 0, \qquad T \geq D \tag{15}$$

This condition occurs at the point where the rate change of specific energy divided by fuel flow $(\mathrm{d}E_s/\mathrm{d}t/w_f)$ is maximum. At each energy level, this expression specifies an optimal altitude. By satisfying the preceding necessary condition at various energy levels within the aircraft flight envelope, a fuel-optimal climb schedule can be constructed in the altitude-airspeed plane.

The minimum-fuel climb schedule was determined by first selecting a set of energy levels. For each specific energy level, with PLA fixed, the optimization toolbox [20] function fmincon was used to find V, α, and γ that maximize the function

$$f = \frac{V(T\cos\alpha - D)}{\dot{m}} \tag{16}$$

with inequality constraints on the sum of the horizontal forces, the sum of the vertical forces, and the flight-path angle

$$\sum F_x \geq 0, \qquad \sum F_z \geq 0, \qquad \gamma \geq 0 \tag{17}$$

and altitude is constrained by the energy equation E_s.

2. Energy-State Method for Maximum-Endurance Descent

In a similar manner, the power-off glide trajectory that maximizes the aircraft endurance can be determined from the energy-state model by satisfying the necessary condition [14]

$$\left.\frac{\partial[V(T-D)]}{\partial h}\right|_{E=\mathrm{Const}} = 0, \qquad T \leq D \tag{18}$$

This condition occurs at the point where the rate change of specific energy (dE_s/dt) is minimum. As in the computation of the fuel-optimal climb path, this necessary condition can be satisfied at various energy levels within the flight envelope to determine an endurance-maximizing glide schedule in the altitude-airspeed plane.

The maximum-endurance descent was computed for two cases: with the throttle at flight idle and with the throttle at zero. In most works on this subject, it is assumed that thrust and fuel flow are zero. However, turning off the engines in flight may not be acceptable in many cases. To compute the schedule, a set of energy levels was chosen. For each specific energy level, the optimization toolbox function fmincon was used to find V, α, and γ that minimize the energy loss rate

$$f = \frac{V(D - T\cos\alpha)}{mg} \tag{19}$$

(the numerator is assumed positive) with an equality constraint on the sum of the vertical forces

$$\sum F_z = 0 \tag{20}$$

an inequality constraint on the flight-path angle

$$\gamma \leq 0 \tag{21}$$

and altitude is constrained by the energy equation, Eq. (12)

C. Periodic Cruise Results, F/A-18

Simulations were performed with military power during the climb phase, and then the throttle was set to either flight idle or zero. The power was reduced when the specified maximum energy level was reached, and a positive flight-path angle was maintained until the speed decreased to the point where the descent schedule was reached. The best angle was found to be around 11.5 deg. The descent schedule was then followed down to minimum altitude. Constant-energy transfer arcs between the climb and descent paths (with intermediate throttle) were found to lower the efficiency. For this aircraft, the steady-state cruise point was found to be $h = 39,360$ ft, $V = 758.5$ ft/s, PLA $= 60.0$, $\alpha = 4.72$ deg, and the fuel flow rate was 3283.7 $\mathrm{lb}_m/\mathrm{h}\boldsymbol{M_A}$.

1. Periodic Cruise with Zero Minimum Throttle

The trajectory for the zero-power descent is shown in Fig. 3. After iterating on several values, it was determined that a maximum specific energy of 35,000 ft gave the most savings. The time of flight was 862.9 s and the mass of fuel consumed was 665.3 lbm, giving an average fuel consumption rate of 2775.6 lbm/h, which is a 15.5% improvement over the steady-state cruise condition. Note that the amount of fuel consumed was 1.9% of the assumed aircraft mass, and so the assumption of constant mass for the analysis seems reasonable. It should be noted that the model does not include windmill drag or inlet drag when the throttle is set to zero, and so the true savings would be less.

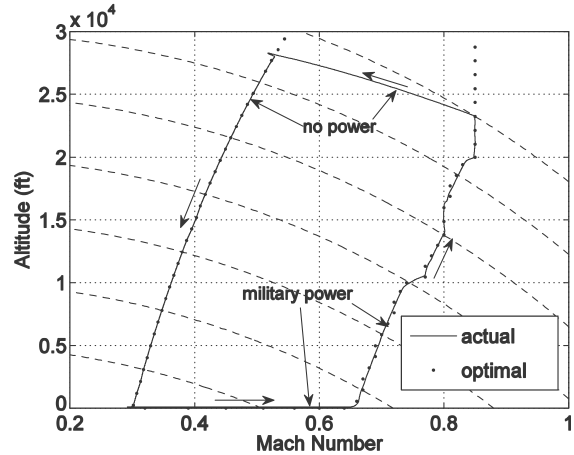

Fig. 3 Trajectory #1: optimal periodic cruise with zero minimum throttle.

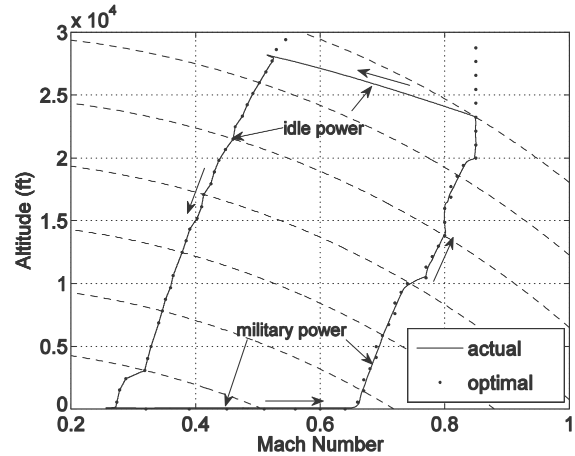

Fig. 4 Trajectory #2: periodic cruise, flight-idle minimum throttle.

2. Periodic Cruise with Idle Minimum Throttle

The results for the idle power case are shown in Fig. 4. The time of flight was 864.3 s, and the total fuel mass consumed was 1004.1 lbm. This means that the average fuel consumption rate was 4182.3 lbm/h, which is a 27.4% loss of efficiency relative to the steady-state cruise point. The time histories are very similar to those of the preceding case, despite the fact that the engines are on throughout the flight. Lowering the maximum specific energy level to 20,000 ft resulted in a 34% loss of efficiency if the transfer arc is flown at minimum thrust and constant γ, and a 29% loss of efficiency if the transfer arc is flown at constant energy.

3. Periodic Cruise with Mixed Minimum Throttle Conditions

Because both engines at idle did not produce any savings, and having both engines shut down is not desirable for practical purposes, the case of only one engine off and one at idle during the descent was investigated. The trajectory is shown in Fig. 5. The time of flight for one period was 860.6 s, and the total fuel consumed was 826.6 lbm, resulting in an average fuel consumption rate of 3459.9 lbm/h, which is a 5.4% loss of efficiency relative to the steady-state cruise point.

4. Improved Transition Arcs

By making some modifications to the transition trajectories between fuel-optimal climb and endurance-maximizing descent, some additional fuel savings were realized for the zero minimum throttle case. In this case the transition between the climb and descent schedules was not at a constant flight-path angle, and there is a dive at the bottom of the descent to trade some altitude for airspeed. The

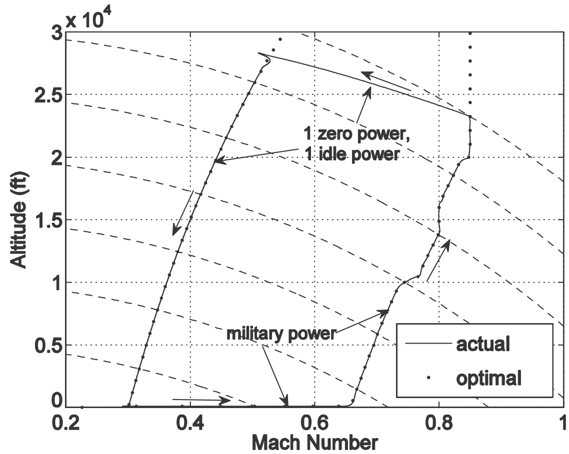

Fig. 5 Trajectory #3: periodic cruise, mixed minimum throttle conditions.

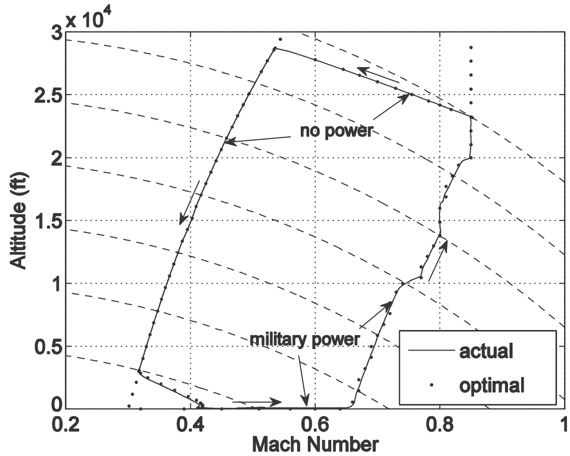

Fig. 6 Trajectory #3: optimal periodic cruise with modified transfer arcs.

controller was altered to follow altitude rate commands during the transfer arcs because this provided better trajectory tracking.

The time of flight in this case was 816.3 s, and the total fuel consumed was 620.2 lbm, giving an average fuel consumption rate of 2735.2 lbm/h, which is a 16.7% improvement over the steady-state cruise condition. The results of the simulation are shown in Fig. 6. The dive at the end of the descent can be seen in Fig. 6. A higher flight-path angle is reached in the high-altitude transfer.

D. Periodic Cruise Results for the F-4 Aircraft Model

The steady-state cruise condition for this model was found to be $h = 17,592.7$ ft, $V = 522.9$ ft/s, $\eta = 0.3339$. The fuel consumption rate for this condition is 3792 lbm/h. The engine model is linear in throttle and it was difficult to determine what the throttle setting for flight idle should be. It was not clear whether the engine model thrust was net or not. It was decided to use a value of 0.1 (10%) for the minimum throttle, which should be a conservative estimate. Several periodic trajectories were constructed using different values for maximum energy. Fuel savings over steady-state cruise were achieved using a periodic trajectory, but the best that could be achieved, by varying the maximum energy, was 2.7% (time of flight = 875.3 s, fuel consumed = 897.2 lbm, average fuel consumption rate = 3690 lbm/h). The results are shown in Fig. 7. The maximum altitude was approximately 24,000 ft, and the minimum altitude was 100 ft.

IV. Hodograph Analysis

In this section, analysis of hodograph sets is used to verify the results of the previous simulations. A hodograph set consists of the

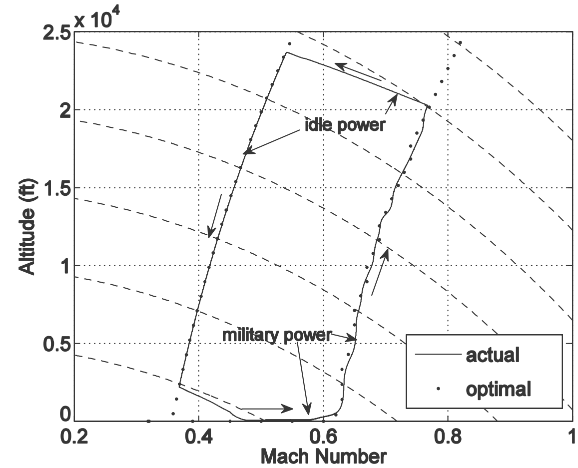

Fig. 7 Periodic trajectory for F-4 model.

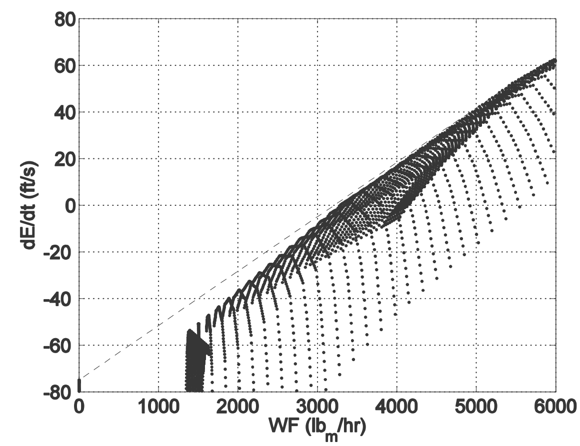

Fig. 8 F/A-18 hodograph set for zero minimum throttle.

set of state rates that can be achieved using admissible controls. For the aircraft endurance study, the hodograph set is two-dimensional with coordinates being the rate of change in specific energy and fuel flow rate. The energy-state model is an approximation of the point-mass model, where energy and mass become the state variables. The hodograph is formed by determining the boundary of possible rates of the state variables E_s and m that can be achieved for admissible values of the control variables. Because the steady-state cruise is performed at constant energy, the corresponding fuel flow rate is given by the value of the fuel flow rate at the point where the hodograph set boundary crosses the zero-energy-rate axis ($\dot{E}_s = 0$, or $E_s =$ constant, implies no acceleration, i.e., the aircraft is in trim). Optimal control theory states that optimal solutions exist only if the hodograph set is convex [13]. If the hodograph set is nonconvex, so that a straight line that is tangent to two points on the hodograph set crosses the zero-energy-rate axis at a lower value of the fuel flow rate than the hodograph set itself, then the existence of a switching control with better cruise performance is indicated [6].

Because the control variables in the energy-state model are altitude and throttle setting, the hodograph sets in the following sections were constructed by selecting a range of altitude points around the cruise point, and a range of throttle points. Airspeed was determined from the energy equation, where the energy is fixed at the cruise point. Every point for which the aircraft can be trimmed in the vertical axis is an admissible point. Enough points were generated to get an idea of where the boundaries of the set were, and the MATLAB® function "convhull" was used to compute the convex hull of the set, which is the smallest convex set that contains all of the points in the original set. If the hodograph is nonconvex, then the convex hull will form the line tangent to the hodograph as described in the preceding paragraph.

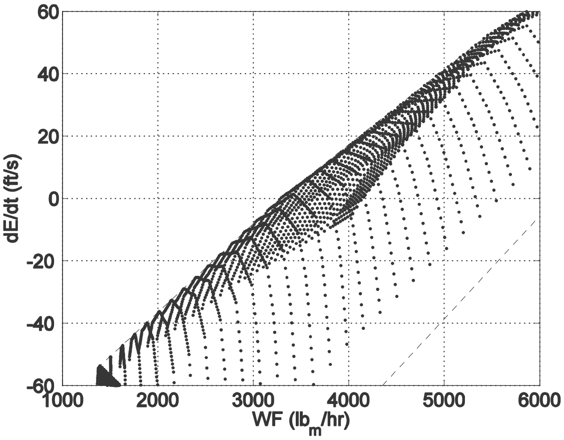

Fig. 9 F/A-18 hodograph set for idle minimum throttle.

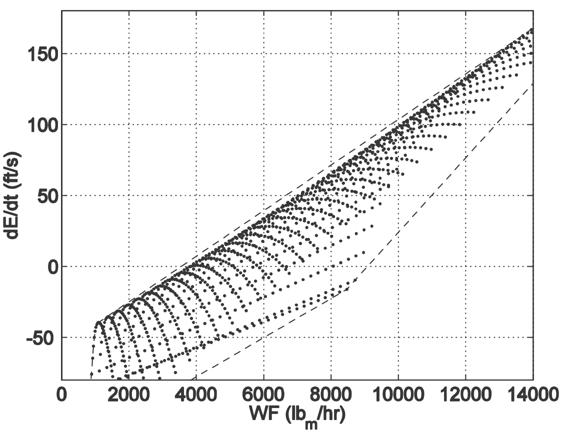

Fig. 10 Hodograph set for F-4 model with 10% minimum throttle.

A. Analysis of the F/A-18 Aircraft Model

Hodograph sets for minimum throttle settings of zero and flight idle are shown in Figs. 8 and 9, respectively. The dashed line is the convex hull of the set. The two graphs are the same except for the points where fuel flow is zero, and the convex hull thus differs. In the first graph, the convex hull can be seen above the boundary of the set in a region around the cruise point ($w_f = 3283.7$, $\dot{E}_s = 0$). In the second graph, the convex hull is indistinguishable because the set is convex, meaning no savings are possible for this aircraft if the minimum throttle setting permitted is flight idle.

B. Hodograph Set Analysis for the F-4 Aircraft

A hodograph set was constructed for the F-4 aircraft using 10% as the minimum value of the throttle and is shown in Fig. 10. The hodograph set shows nonconvexity, indicating that fuel savings are possible. Lower minimum throttle setting may permit better fuel savings.

V. Conclusions

This paper showed that periodic trajectories could be constructed without solving a two-point boundary-value problem that achieve greater endurance than the steady-state solution. Periodic cruise trajectories can be constructed using energy-based methods by patching together a fuel-optimal climb schedule with a maximum-endurance glide path, with minimum-energy-loss transition arcs. Two different aircraft models produced significantly different results, but the fidelity of the two models was also significantly different. Also, one aircraft was powered by turbojets and the other by turbofans, which could also be part of the reason for the differences. Although it had been conjectured in the literature that large improvements in endurance could be achieved with a periodic trajectory, at best only modest gains were achieved in this study. However, the trajectories in this study were suboptimal, and the nearness to optimality needs to be evaluated. Other types of aircraft, such as those with a higher lift-to-drag ratio or other types of engines, might have different results.

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